Higher–Order Corrections to the Effective Gravitational Action from Noether Symmetry Approach

S. Capozziello[*] and G. Lambiase[†]
Dipartimento di Scienze Fisiche "E.R. Caianiello"
Istituto Nazionale di Fisica Nucleare, Sez. di Napoli,
Universitá di Salerno, 84081 Baronissi (Sa), Italy.

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Abstract

Higher–order corrections of Einstein–Hilbert action of general relativity can be recovered by imposing the existence of a Noether symmetry to a class of theories of gravity where Ricci scalar R and its d'Alembertian $\Box R$ are present. In several cases, it is possible to get exact cosmological solutions or, at least, to simplify dynamics by recovering constants of motion. The main result is that a Noether vector seems to rule the presence of higher–order corrections of gravitaty.

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[*]E-mail: capozziello@vaxsa.csied.unisa.it, capozziello@na.infn.it
[†]E-mail: lambiase@physics.unisa.it

1 Introduction

The issue of recovering a suitable effective action seems to be one of the main ways to construct a consistent theory of quantum gravity [1]. Starting from pioneering works of Sakharov [2], the effects of vacuum polarization on the gravitational constant, i.e. the fact that gravitational constant can be induced by vacuum polarization, have been extensively investigated. All these attempts led to take into account gravitational actions extended beyond the simple Einstein–Hilbert action of general relativity which is linear in the Ricci scalar R.

At the beginning, the motivation was to investigate alternative theories in order to see if gravitational effects could be recovered in any case.

The Brans–Dicke approach is one of this attempt which, asking for dynamically inducing the gravitational coupling by a scalar field, is more coherent with the Mach principle requests [3].

Besides, it has been realized that corrective terms are inescapable if we want to obtain the effective action of quantum gravity on scales close to the Planck length (see e.g. [4]). In other words, it seems that, in order to construct a renormalizable theory of gravity, we need higher–order terms of curvature invariants such as $R^2, R^{\mu\nu}R_{\mu\nu}, R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}, R\Box R$, $R\Box^k R$ or nonminimally coupled terms between scalar fields and geometry as $\varphi^2 R$.

Stelle [5] constructed a renormalizable theory of gravity by introducing quadratic terms in curvature invariants. Barth and Christensen gave a detailed analysis of the one–loop divergences of fourth–order gravity theories giving the first general scheme of quantization of higher–order theories [6, 7]. Several results followed and today it is well known that a renormalizable theory of gravity is obtained, at one–loop level, if at least quadratic terms in the Riemann curvature tensor and its contractions are introduced [1]. Any action, where a finite number of terms involving power laws of curvature tensor or its derivatives appears, is a low–energy approximation to some fundamental theory of gravity which, up to now, is not available. For example, string theory or supergravity present low–energy effective actions where higher–order or nonminimally coupled terms appear. [8].

However, if Lagrangians with higher–order terms or arbitrary derivatives in curvature invariants are considered, they are expected to be non–local and give rise to some characteristic length l_0 of the order of Planck length. The expansion in terms of R and $\Box R$, for example, at scales larger than l_0 produces infinite series which should break near l_0 [9].

With these facts in mind, taking into account such Lagrangians, means to make further steps toward a complete renormalizable theory of gravity. For instance, Vilkovisky [4] considered a non–local Lagrangian of the form $Rf(\Box)R$, where

$$f(\Box) = \int \frac{\rho(x)}{\Box - x} dx \,, \tag{1.1}$$

in order to construct an effective action of quantum gravity. Immediately, one realizes

that it can be approximated by the sum

$$\sum_{i=0}^{k} R\Box^i R\,, \tag{1.2}$$

so that we get $(2k+4)$–order field equations. Also the case $R\frac{1}{\Box}R$ has to be mentioned since it can be regarded as a conformal anomaly [10].

We have to do an important remark at this point. In the effective action, terms as $R^j\Box^i R$ have to be taken into account since pure terms as $\Box^i R$ are total divergences and can be ignored. Terms as $\Box^i R\Box^j R$ can be integrated by parts giving $R\Box^{i+j}R$ [11, 12, 13].

We are restricting to Lagrangians containing Ricci scalar and its derivative since in this case it is quite straightforward to obtain conformal transformations relating higher–order gravity theories to general relativity with a certain number of scalar fields [14].

Furthermore, only these Lagrangians are interesting for constructing cosmological models (at least if we restrict the discussion to homogeneous cosmological models).

However, we have to consider the fact that in the limit of the classical theory, higher–order Lagrangians give rise to superfluous degrees of freedom. This is a controversy in literature [15, 16, 17], which, in our knowledge is not solved. Some authors [18], discuss the possibility that only solutions which are expandable in powers of $\hbar$ are *self–consistent*, others [19] consider such superfluous degrees of freedom as phases of oscillations around the Friedman behaviour [20]. In any case, the transition to our classical observed universe has to be accurately discussed.

In this paper, we are going to discuss if such extra–terms in the effective gravitational action can be recovered by asking for symmetries of a Lagrangian which generic form is

$$\mathcal{L} = \sqrt{-g}\,F(R, \Box R)\,. \tag{1.3}$$

We are using the so called *Noether Symmetry Approach* which was extensively used to study nonminimally coupled theories of the form

$$\mathcal{L} = \sqrt{-g}\left[F(\varphi)R + \frac{1}{2}\nabla_\mu\varphi\nabla^\mu\varphi - V(\varphi)\right]\,, \tag{1.4}$$

where ∇_μ is the covariant derivative. In [21, 22, 23, 24], it was shown that asking for the existence of a Noether symmetry

$$L_X\mathcal{L} = 0 \to X\mathcal{L} = 0\,, \tag{1.5}$$

where L_X is the Lie derivative with respect to the Noether vector X, it is possible to select physically interesting forms of the interaction potential $V(\varphi)$ and the gravitational coupling $F(\varphi)$. The scalar field φ is generic and it can represent the matter counterpart in an early universe dynamics.

The existence of Noether symmetries allows to select constants of motion so that the dynamics results simplified. Often such a dynamics is exactly solvable by a straightforward change of variables where a cyclic one is present.

Here we want to apply the same method to higher–order theories of the form (1.3). In particular, we take into account suitable minisuperspaces whose degrees of freedom are a, the scalar factor of the universe, R, the Ricci scalar, and $\Box R$ the d'Alembertian of Ricci scalar, related among them by some Lagrange constraints. As we shall see in next section, using a Friedman–Robertson–Walker (FRW) metric it is possible to reduce the Lagrangian (1.3) to a point–like one and then apply the Noether technique. The main result is that several fourth, sixth and eight–order interesting Lagrangians are recovered by asking for the Noether symmetry. For example, a term as $\sqrt{R\Box R}$, which is a part of the so–called a_3 anomaly [25, 26] is connected to the existence of the Noether symmetry.

A similar result works for $F_0 R^{3/2}$ which is related to the Liouville field theory by a conformal transformation. The same technique gives Lagrangians like

$$\mathcal{L} = \sqrt{-g}\,(F_1 R + F_2 R^2)\,, \qquad \text{or} \quad \mathcal{L} = \sqrt{-g}\,(F_1 R + F_2 R^2 + F_3 R\Box R)\,,$$

where F_i are constants, widely studied in literature, e.g. [17, 20, 27, 28, 29, 30, 31]. In conclusion, it seems that the Noether approach is related to the recovering of one–loop and trace anomaly corrections of quantum gravity.

The paper is organized as follows. Sect.2 is devoted to the discussion of the generic point–like Lagrangian which can be recovered from theories like (1.3). In Sect.3, we apply Noether vector to such a Lagrangian showing that the existence of a symmetry selects its form. Fourth–order and higher than fourth–order models are discussed in Sect.4 and 5, respectively. In both cases we study the related FRW cosmology. Discussion and conclusions are drawn in Sect.6.

2 Higher-Order Point-like Lagrangians and Equations of Motion

A generic higher–order theory in four dimensions can be described by the action

$$\mathcal{A} = \int d^4x\sqrt{-g}F(R, \Box R, \Box^2 R, \ldots, \Box^k R)\,. \tag{2.1}$$

We are using physical units $8\pi G_N = c = \hbar = 1$. Equations of motion can be deduced by the method worked out in [13, 32]

$$\begin{aligned} G^{\mu\nu} &= \frac{1}{\mathcal{G}}\left\{\frac{1}{2}g^{\mu\nu}(F - \mathcal{G}R) + (g^{\mu\lambda}g^{\nu\sigma} - g^{\mu\nu}g^{\lambda\sigma})\mathcal{G}_{;\,\lambda\sigma} + \right. \\ &+ \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{i}(g^{\mu\nu}g^{\lambda\sigma} + g^{\mu\lambda}g^{\nu\sigma})(\Box^{j-i})_{;\,\sigma}\left(\Box^{i-j}\frac{\partial F}{\partial\Box^i R}\right)_{;\,\lambda} + \\ &- \left. g^{\mu\nu}g^{\lambda\sigma}\left[(\Box^{j-1})_{;\,\sigma}\Box^{i-j}\frac{\partial F}{\partial\Box^i R}\right]\right\}\,, \end{aligned} \tag{2.2}$$

where

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\,, \tag{2.3}$$

is the Einstein tensor, and

$$\mathcal{G}=\sum_{j=0}^{k}\Box^{j}\left(\frac{\partial F}{\partial\Box^{j}R}\right)\,. \tag{2.4}$$

As we said, these are pure gravity $(2k+4)$–order field equations. Matter can be taken into account by introducing the stress–energy tensor $T^{\mu\nu}$ of a (non)minimally coupled scalar field [11, 33].

For the sake of simplicity, let us restrict to the Lagrangian (1.3). In this case, we have eight–order field equations which becomes of sixth–order if the theories is linear in $\Box R$. To apply Noether Symmetry Approach, let us take into account the point–like FRW Lagrangian

$$\mathcal{L}=\mathcal{L}(a,\dot{a},R,\dot{R},\Box R,(\dot{\Box R}))\,. \tag{2.5}$$

The variables R and $\Box R$ can be considered independent and, by the method of Lagrange multipliers, we can eliminate higher than one time derivatives (for fourth–order case see e.g. [34]). The action related to Lagrangian (1.3) becomes

$$\mathcal{A}=2\pi^{2}\int dt\left\{a^{3}F-\lambda_{1}\left[R+6\left(\frac{\ddot{a}}{a}+\left(\frac{\dot{a}}{a}\right)^{2}+\frac{k}{a^{2}}\right)\right]-\lambda_{2}\left[\Box R-\ddot{R}-3\left(\frac{\dot{a}}{a}\right)\dot{R}\right]\right\}\,. \tag{2.6}$$

$\lambda_{1,2}$ are given by varying the action with respect to R and $\Box R$, that is

$$\lambda_{1}=a^{3}\,\frac{\partial F}{\partial R}\,,\quad \lambda_{2}=a^{3}\frac{\partial F}{\partial(\Box R)}\,. \tag{2.7}$$

After an integration by parts, the (Helmholtz type) point–like Lagrangian is

$$\mathcal{L}=6a\dot{a}^{2}\frac{\partial F}{\partial R}+6a^{2}\dot{a}\frac{d}{dt}\left(\frac{\partial F}{\partial R}\right)-a^{3}\dot{R}\frac{d}{dt}\left(\frac{\partial F}{\partial(\Box R)}\right)+a^{3}\left[F-\left(R+\frac{6k}{a^{2}}\right)\frac{\partial F}{\partial R}-\Box R\,\frac{\partial F}{\partial(\Box R)}\right]\,. \tag{2.8}$$

A remark is necessary at this point. We can also take into account

$$\lambda_{1}=a^{3}\left[\frac{\partial F}{\partial R}+\Box\frac{\partial F}{\partial(\Box R)}\right]\,, \tag{2.9}$$

as a Lagrange multiplier [33]. The Lagrangian which comes out differs from (2.8) just for a term vanishing on the constraint, being

$$\tilde{\mathcal{L}}=\mathcal{L}-a^{3}\left\{R+6\left[\frac{\ddot{a}}{a}+\left(\frac{\dot{a}}{a}\right)^{2}+\frac{k}{a^{2}}\right]\right\}\Box\frac{\partial F}{\partial(\Box R)}\,. \tag{2.10}$$

¿From this point of view, considering the point–like Lagrangian $\mathcal{L}$ or $\tilde{\mathcal{L}}$ is completely equivalent (this remark is obvious dealing with the equations of motion).

Let us now derive the Euler–Lagrangian equations from (2.8). They can be also deduced from the Einstein equations (2.2). For the sake of clarity, let us derive them step by step. The equation

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{a}} = \frac{\partial \mathcal{L}}{\partial a} \tag{2.11}$$

gives

$$\begin{aligned} &\left[R\frac{\partial F}{\partial R} + \Box R\frac{\partial F}{\partial(\Box R)} - F\right] + 2\left[3H^2 + 2\dot{H} + \frac{k}{a^2}\right]\frac{\partial F}{\partial R} + \\ + &\ \ 2\left[\Box R - H\dot{R}\right]\frac{\partial^2 F}{\partial R^2} + \dot{R}(\dot{\Box R})\frac{\partial^2 F}{\partial(\Box R)^2} + \left[2\Box^2 R - 2H(\dot{\Box R}) + \dot{R}^2\right]\frac{\partial^2 F}{\partial R\partial(\Box R)} + \\ + &\ \ 2\dot{R}^2\frac{\partial^3 F}{\partial R^3} + 2(\dot{\Box R})^2\frac{\partial^3 F}{\partial R\partial(\Box R)^2} + 4\dot{R}(\dot{\Box R})\frac{\partial^3 F}{\partial R^2\partial(\Box R)} = 0\,. \end{aligned} \tag{2.12}$$

The equation

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{R}} = \frac{\partial \mathcal{L}}{\partial R} \tag{2.13}$$

gives

$$\Box\frac{\partial \mathcal{F}}{\partial(\Box R)} = 0\,. \tag{2.14}$$

Finally,

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial(\dot{\Box R})} = \frac{\partial \mathcal{L}}{\partial \Box R}\,, \tag{2.15}$$

coincides with the Lagrange constraints

$$\Box R = \ddot{R} + 3H\dot{R}\,, \tag{2.16}$$

$$R = -6\left(\dot{H} + 2H^2 + \frac{k}{a^2}\right)\,. \tag{2.17}$$

Here $H = \dot{a}/a$ is the Hubble parameter. The condition on the energy

$$E_{\mathcal{L}} \equiv \dot{a}\frac{\partial \mathcal{L}}{\partial \dot{a}} + \dot{R}\frac{\partial \mathcal{L}}{\partial \dot{R}} + (\dot{\Box R})\frac{\partial \mathcal{L}}{\partial(\dot{\Box R})} - \mathcal{L} = 0\,, \tag{2.18}$$

which is the $(0,0)$–Einstein equation, gives

$$H^2\left(\frac{\partial F}{\partial R}\right) + H\frac{d}{dt}\left(\frac{\partial F}{\partial R}\right) + \frac{\Gamma}{6} = 0\,, \tag{2.19}$$

where

$$\Gamma = \left(R + \frac{6k}{a^2}\right)\frac{\partial F}{\partial R} + \Box R\frac{\partial F}{\partial(\Box R)} - F - \dot{R}\frac{d}{dt}\left(\frac{\partial F}{\partial(\Box R)}\right)\,, \tag{2.20}$$

can be interpreted as a sort of effective density (see also [33]).

3 Noether Symmetry Approach

A Noether symmetry for the Lagrangian (2.8) exists if the condition (1.5) holds. It is nothing else but the contraction of a Noether vector X, defined on the tangent space $TQ = \{q_i, \dot{q}_i\}$ of the Lagrangian $\mathcal{L} = \mathcal{L}(q_i, \dot{q}_i)$, with the Cartan one–form, generically defined as

$$\theta_{\mathcal{L}} \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}_i} dq^i \,. \tag{3.1}$$

Condition (1.5) gives

$$i_X \theta_{\mathcal{L}} = \Sigma_0 \,, \tag{3.2}$$

where i_X is the inner derivative and Σ_0 is the conserved quantity [21, 22, 23, 35, 36]. In other words, the existence of the symmetry is connected to the existence of a vector field

$$X = \alpha^i(q) \frac{\partial}{\partial q^i} + \frac{d\alpha^i(q)}{dt} \frac{\partial}{\partial \dot{q}^i} \,, \tag{3.3}$$

where at least one of the components $\alpha^i(q)$ have to be different from zero. In our case, the tangent space is

$$TQ = \{a, \dot{a}, R, \dot{R}, \Box R, (\dot{\Box R})\} \,, \tag{3.4}$$

and the generator of symmetry is

$$X = \alpha \frac{\partial}{\partial a} + \beta \frac{\partial}{\partial R} + \gamma \frac{\partial}{\partial(\Box R)} + \dot{\alpha} \frac{\partial}{\partial \dot{a}} + \dot{\beta} \frac{\partial}{\partial \dot{R}} + \dot{\gamma} \frac{\partial}{\partial (\dot{\Box R})} \,. \tag{3.5}$$

The functions α, β, γ depend on the variables $a, R, \Box R$. A Noether symmetry exists if at least one of them is different from zero. Their analytic forms can be found by expliciting (1.5), which corresponds to a set of $1 + \frac{n(n+1)}{2}$ partial differential equations given by equating to zero the terms in $\dot{a}^2$, $\dot{R}^2$, $(\dot{\Box R})^2$, $\dot{a}\dot{R}$ and so on. In our specific case, $n = 3$ is the dimension of the configuration space. We get a system of seven partial differential equations

$$\frac{\partial F}{\partial R}\left(\alpha + 2a\frac{\partial \alpha}{\partial a}\right) + a\frac{\partial^2 F}{\partial R^2}\left(\beta + a\frac{\partial \beta}{\partial a}\right) + a\frac{\partial^2 F}{\partial R \partial(\Box R)}\left(\gamma + a\frac{\partial \gamma}{\partial a}\right) = 0 \,, \tag{3.6}$$

$$-6\frac{\partial^2 F}{\partial R^2}\frac{\partial \alpha}{\partial R} + \frac{\partial^2 F}{\partial R \partial(\Box R)}\left(3\alpha + 2a\frac{\partial \beta}{\partial R}\right) + \beta a\frac{\partial^3 F}{\partial R^2 \partial(\Box R)} + \gamma a\frac{\partial^3 F}{\partial R \partial(\Box R)^2} + a\frac{\partial \gamma}{\partial R}\frac{\partial^2 F}{\partial(\Box R)^2} = 0 \,, \tag{3.7}$$

$$6\frac{\partial^2 F}{\partial R \partial(\Box R)}\frac{\partial \alpha}{\partial(\Box R)} - a\frac{\partial^2 F}{\partial(\Box R)^2}\frac{\partial \beta}{\partial(\Box R)} = 0 \,, \tag{3.8}$$

$$12\frac{\partial F}{\partial R}\frac{\partial \alpha}{\partial R} + 6\frac{\partial^2 F}{\partial R^2}\left(2\alpha + a\frac{\partial \alpha}{\partial a} + a\frac{\partial \beta}{\partial R}\right) + a\frac{\partial^2 F}{\partial R \partial(\Box R)}\left(6\frac{\partial \gamma}{\partial R} - 2a\frac{\partial \beta}{\partial a}\right) -$$

$$
-a^2\frac{\partial\gamma}{\partial a}\frac{\partial^2 F}{\partial(\Box R)^2}+6\beta a\frac{\partial^3 F}{\partial R^3}+6\gamma a\frac{\partial^3 F}{\partial R^2\partial(\Box R)}=0\,, \tag{3.9}
$$

$$
\begin{aligned}
&12\frac{\partial F}{\partial R}\frac{\partial\alpha}{\partial(\Box R)}+\frac{\partial^2 F}{\partial R\partial(\Box R)}\left(12\alpha+6a\frac{\partial\alpha}{\partial a}+6a\frac{\partial\gamma}{\partial(\Box R)}\right)+6a\frac{\partial^2 F}{\partial R^2}\frac{\partial\beta}{\partial(\Box R)}-\\
&\quad -a^2\frac{\partial^2 F}{\partial(\Box R)^2}\frac{\partial\beta}{\partial a}+6\gamma a\frac{\partial^3 F}{\partial R\partial(\Box R)^2}+6\beta a\frac{\partial^3 F}{\partial R^2\partial(\Box R)}=0\,,
\end{aligned} \tag{3.10}
$$

$$
\begin{aligned}
&\frac{\partial^2 F}{\partial(\Box R)^2}\left(3\alpha+a\frac{\partial\gamma}{\partial(\Box R)}+a\frac{\partial\beta}{\partial R}\right)+\frac{\partial^2 F}{\partial R\partial(\Box R)}\left(2a\frac{\partial\beta}{\partial(\Box R)}-6\frac{\partial\alpha}{\partial R}\right)-6\frac{\partial^2 F}{\partial R^2}\frac{\partial\alpha}{\partial\Box R}+\\
&\quad +\beta a\frac{\partial^3 F}{\partial R\partial(\Box R)^2}+\gamma a\frac{\partial^3 F}{\partial(\Box R)^3}=0\,,
\end{aligned} \tag{3.11}
$$

$$
\begin{aligned}
&3\alpha\left(F-R\frac{\partial F}{\partial R}\right)-\beta aR\frac{\partial^2 F}{\partial R^2}-3\alpha\Box R\frac{\partial F}{\partial(\Box R)}-\gamma a\Box R\frac{\partial^2 F}{\partial(\Box R)^2}-a\frac{\partial^2 F}{\partial R\partial(\Box R)}(\beta\Box R+\gamma R)-\\
&\quad -\frac{6k}{a^2}\left[\alpha\frac{\partial F}{\partial R}+\beta a\frac{\partial^2 F}{\partial R^2}+\gamma a\frac{\partial^2 F}{\partial R\partial(\Box R)}\right]=0\,.
\end{aligned} \tag{3.12}
$$

The system is overdetermined and, if solvable, enables one to assign α,β,γ, and $F(R,\Box R)$. In such a case, we can always transform the Lagrangian (2.8) so that

$$
\mathcal{L}(a,\dot a,R,\dot R,\Box R,(\dot{\Box R}))\to\mathcal{L}(u,\dot u,w,\dot w,\dot z)\,, \tag{3.13}
$$

where z is a cyclic variable and the dynamics is simplified. This change of variables can be easily obtained by the conditions

$$
\begin{aligned}
i_X dz &= \alpha\frac{\partial z}{\partial a}+\beta\frac{\partial z}{\partial R}+\gamma\frac{\partial z}{\partial\Box R}=1\,, &(3.14)\\
i_X dw &= \alpha\frac{\partial w}{\partial a}+\beta\frac{\partial w}{\partial R}+\gamma\frac{\partial w}{\partial\Box R}=0\,, &(3.15)\\
i_X du &= \alpha\frac{\partial u}{\partial a}+\beta\frac{\partial u}{\partial R}+\gamma\frac{\partial u}{\partial\Box R}=0\,, &(3.16)
\end{aligned}
$$

which strictly depend on the form of α,β,γ. Once we solve the dynamics in the system $\{z,w,u\}$, by the inverse transformation

$$
\{z(t),w(t),u(t)\}\to\{a(t),R(t),\Box R(t)\}\,, \tag{3.17}
$$

we recover the dynamics in our primitive physical variables. Here t is the cosmic time.

However, we have to stress that we are considering a constrained dynamics since the variable a, R, $\Box R$ are related each other. In next sections, we show that the existence of the Noether symmetry, i.e. solving the system (3.6)–(3.12), gives models of physical interest.

4 Fourth–Order Gravity

If the Lagrangian (2.8) does not depend on $\Box R$, we are dealing with fourth–order equations of motion. Furthermore, if $F(R) = R + 2\Lambda$, the standard second–order gravity is recovered, being Λ the cosmological constant.

The configuration space is two-dimensional so that system (3.6)–(3.12) reduces to

$$\frac{dF}{dR}\left(\alpha + 2a\frac{\partial\alpha}{\partial a}\right) + a\left(\frac{d^2F}{dR^2}\right)\left(\beta + a\frac{\partial\beta}{\partial a}\right) = 0\,, \tag{4.1}$$

$$a^2\left(\frac{d^2F}{dR^2}\right)\frac{\partial\alpha}{\partial R} = 0\,, \tag{4.2}$$

$$2\left(\frac{dF}{dR}\right)\frac{\partial\alpha}{\partial R} + \frac{d^2F}{dR^2}\left(2\alpha + a\frac{\partial\alpha}{\partial a} + a\frac{\partial\beta}{\partial R}\right) + a\beta\frac{d^3F}{dR^3} = 0\,, \tag{4.3}$$

$$3\alpha\left(F - R\frac{dF}{dR}\right) - a\beta R\frac{d^2F}{dR^2} - \frac{6k}{a^2}\left(\alpha\frac{dF}{dR} + a\beta\frac{d^2F}{dR^2}\right) = 0\,, \tag{4.4}$$

where we need only four equations. Immediately we see that a Noether symmetry exists for $F(R) = R + 2\Lambda$. Discarding this trivial case, we get the solution

$$\alpha = \frac{\beta_0}{a}\,, \quad \beta = -2\beta_0\frac{R}{a^2}\,, \quad F(R) = F_0 R^{3/2}\,, \tag{4.5}$$

for any value of the spatial curvature constant $k = 0, \pm 1$; β_0 and F_0 are constants. This case is interesting in conformal transformations from Jordan frame to Einstein frame [37, 38] since it is possible to give explicit form of scalar field potential. In fact, if

$$\tilde{g}_{\alpha\beta} \equiv \left(\frac{dF}{dR}\right) g_{\alpha\beta}\,, \quad \varphi = \sqrt{\frac{3}{2}}\ln\left(\frac{dF}{dR}\right)\,, \tag{4.6}$$

we have the conformal equivalence of the Lagrangians

$$\mathcal{L} = \sqrt{-g}\,F_0 R^{3/2} \longleftrightarrow \tilde{\mathcal{L}} = \sqrt{-\tilde{g}}\left[-\frac{\tilde{R}}{2} + \frac{1}{2}\nabla_\mu\varphi\nabla^\mu\varphi - V_0\exp\left(\sqrt{\frac{2}{3}}\varphi\right)\right]\,, \tag{4.7}$$

in our physical units. This is the so–called Liouville field theory and it is one of the few cases where a fourth–order Lagrangian can be expressed, in the Einstein frame, in terms of elementary functions under a conformal transformation. Using Eqs. (3.14) and (3.15), we get the new variables

$$w = a^2 R\,, \quad z = \frac{a^2}{2\beta_0}\,, \tag{4.8}$$

from which Lagrangian (2.8) (without terms in $\Box R$) becomes

$$\mathcal{L} = \frac{9\beta_0}{2}\frac{\dot{z}\dot{w}}{\sqrt{w}} - 9k\sqrt{w} - \frac{1}{2}\sqrt{w^3}\,. \tag{4.9}$$

A further change of variable $y = \sqrt{w}$ gives

$$\mathcal{L} = 9\beta_0 \dot{z}\dot{y} - 9ky - \frac{y^3}{2}\,, \tag{4.10}$$

where z is cyclic. The dynamics is given by the equations

$$\dot{y} = 0\,, \tag{4.11}$$
$$9\beta_0 \ddot{z} + 9k + \frac{3}{2}\,y^2 = 0\,, \tag{4.12}$$
$$9\beta_0 \dot{z}\dot{y} + 9ky + \frac{y^3}{2} = 0\,, \tag{4.13}$$

whose general solution is

$$y(t) \;=\; \Sigma_0 t + y_0\,, \tag{4.14}$$
$$z(t) \;=\; c_4 t^4 + c_3 t^3 + c_2 t^2 + c_1 t + c_0\,. \tag{4.15}$$

Going back to physical variables, we get the cosmological solution

$$a(t) = \sqrt{2\beta_0}\,[c_4 t^4 + c_3 t^3 + c_2 t^2 + c_1 t + c_0]^{1/2}\,, \tag{4.16}$$

$$R(t) = \frac{(\Sigma_0 t + y_0)^2}{2\beta_0[c_4 t^4 + c_3 t^3 + c_2 t^2 + c_1 t + c_0]}\,. \tag{4.17}$$

The constants c_i are combinations of the initial conditions. Their values determine the type of cosmological evolution. For example, $c_4 \neq 0$ gives a power law inflation while, if the regime is dominated by the linear term in c_1, we get a radiation–dominated stage.

The system (4.1)–(4.4) admits other solutions. Another interesting one is given by

$$\alpha = 0\,, \quad \beta = \frac{\beta_0}{a}\,, \quad F(R) = F_1 R + F_2 R^2\,, \tag{4.18}$$

if the condition

$$R = -\frac{6k}{a^2} \tag{4.19}$$

is satisfied. Immediately we get the cosmological solution

$$a(t) = a_0 t^{1/2}\,. \tag{4.20}$$

A similar situation works any time that $d^2F/dR^2 \equiv F'' \neq 0$. The system (4.1)–(4.4) is solved by

$$\alpha = 0\,, \quad \beta = \frac{\beta_0}{aF''}\,. \tag{4.21}$$

Condition (4.19) has to be satisfied and the radiative cosmological solution (4.20) is recovered. Particularly interesting, from the point of view of one–loop corrections of quantum gravity, are polynomial Lagrangian of the form

$$F(R)=\sum_{j=0}^{N}F_jR^j\,,\quad N\geq 0\,,\tag{4.22}$$

where F_j are constant coefficients whose physical dimension depends on j. Cosmological models coming from such theories have been widely studied (see e.g. [39, 40]). However, standard Einstein coupling is recovered if $F_1=-1/2$.

Results are summarized in Table I.

5 Sixth and Eighth–Order Gravity

Considering $\Box R$ as a degree of freedom means to take into account the whole system (3.6)–(3.12) in order to find some Noether symmetry. If $F(R,\Box R)$ depends only linearly on $\Box R$, we have a sixth–order theory, otherwise we are dealing with eighth–order theories.

A simple sixth–order solution of Noether system (3.6)–(3.12) is recovered if

$$\alpha=\frac{\alpha_0}{\sqrt{a}}\,,\quad \beta\ \gamma\ \text{any}\,,\quad F(R,\Box R)=F_1R+F_2\Box R\,,\quad k=0\,.\tag{5.1}$$

Due to the above considerations on pure divergence [11, 12], this theory reduces to the Einstein one where standard cosmological solutions are recovered.

For powers of $\Box R$, we have Noether symmetries given by

$$\alpha=0\,,\quad \beta=\beta_0\,,\quad \gamma=0\,,\quad F(R,\Box R)=F_1R+F_2(\Box R)^n\,,\, n\geq 2\,.\tag{5.2}$$

However, the theory can assume different forms integrating by parts [11, 12, 13]. The equations of motion are

$$4F_1\dot{H}+6F_1H^2+2F_1\frac{k}{a^2}-F_2(1-n)(\Box R)^n-\left(\frac{\dot{R}}{a}\right)\Sigma_0=0\,,\tag{5.3}$$

$$-F_2n(n-1)a^3(\Box R)^{n-2}(\dot{\Box R})=\Sigma_0\,,\tag{5.4}$$

$$\Box R-\ddot{R}-3H\dot{R}=0\,,\tag{5.5}$$

$$6F_1\left[H^2+\frac{k}{a^2}\right]-F_2(1-n)(\Box R)^n+\dot{R}\Sigma_0=0\,.\tag{5.6}$$

Σ_0 is the Noether constant and R is the cyclic variable. The standard Newton coupling is recovered, as usual, for $F_1=-1/2$. A solution for this system is

$$a(t)=a_0t\,,\tag{5.7}$$

for $k=-1$, $\Sigma_0=0$, and for arbitrary n, F_2. Another solution is

$$a(t)=a_0t^{1/2}\,,\tag{5.8}$$

for $k=0$, $\Sigma_0=0$, $F_1=0$ and for arbitrary n, F_2. Finally we get

$$a(t) = a_0 \exp(k_0 t)\,, \tag{5.9}$$

for $k=0$, $\Sigma_0=0$, $F_1=0$.

The radiative cosmological solution (4.20) with the condition (4.19) is recovered for the cases

$$\alpha = 0\,, \quad \beta = \frac{\beta_0}{a}\,, \quad \gamma \text{ any}\,, \quad F(R, \Box R) = F_1 R + F_2 R^2 + F_3 \Box R\,, \tag{5.10}$$

and

$$\alpha = 0\,, \quad \beta = 0\,, \quad \gamma = \frac{\gamma_0}{a}\,, \quad F(R, \Box R) = F_1 R + F_2 R^2 + F_3 R \Box R\,, \tag{5.11}$$

with γ_0 constant. The second one is of physical interest and the related cosmological models have been widely studied [14, 30].

Another Noether symmetry is recovered for

$$\alpha = 0\,, \quad \beta = \beta_0\,, \quad \gamma = \beta_0 \frac{\Box R}{R}\,, \quad F(R, \Box R) = F_1 R + F_2 \sqrt{R \Box R}\,, \tag{5.12}$$

or simply for

$$F(R, \Box R) = F_2 \sqrt{R \Box R}\,. \tag{5.13}$$

This case deserves a lot of attention since $\sqrt{R \Box R}$ is exactly a part of a_3–anomaly [14] which can be recovered from the general analysis of one–loop contributions to gravitational action [25, 26]. Such a fact is indicative since it seems that searching for Noether symmetries could be a relevant method for constructing an effective action of quantum gravity. A straightforward change of variables, given by (3.14)–(3.16), is

$$z = R\,, \quad u = \sqrt{\frac{\Box R}{R}}\,, \quad w = a\,. \tag{5.14}$$

Choosing the standard Einstein coupling $F_1 = -1/2$ the Lagrangian (2.8) becomes

$$\mathcal{L} = 3[w\dot{w}^2 - kw] - F_2 \left[3w\dot{w}^2 u + 3w^2 \dot{w}\dot{u} + \frac{w^3 \dot{z}\dot{u}}{2u^2} - 3kwu\right]\,, \tag{5.15}$$

where z (i.e. R) is the cyclic variable. ¿From (5.15), it is immediate to derive the equations of motion. As above, it is possible to recover the particular solutions

$$a(t) = a_0 t\,, \quad a(t) = a_0 t^{1/2}\,, \quad a(t) = a_0 \exp(k_0 t) \tag{5.16}$$

depending on the set of parameters $\{\Sigma_0, k, F_2\}$. A phase–space view and conformal analysis, as that in [30], gives the conditions for he onset and duration of inflation which, specifically, depends on the sign and the value of F_2 and Σ_0. This fact restricts the set of initial conditions capable of furnishing satisfactory inflationary cosmology as discussed in [14, 30]. In Table II we report the main results of this section.

6 Discussion and Conclusions

In this paper, we have used the Noether Symmetry Approach in order to study higher–order theories of gravity. The existence of a symmetry selects the form of higher–order Lagrangian as in nonminimally coupled theories (1.4); there this technique allows to assign the form of the coupling $F(\varphi)$ and the potential $V(\varphi)$ [21, 22]. Here we discussed theories up to eight–order but it is clear that the method works also for orders beyond. The scheme is always the same: *i*) if a symmetry exists, *ii*) the form of the effective Lagrangian is assigned, *iii*) a suitable change of variables allows to write the dynamics so that a cyclic coordinate appears. The solution of the cosmological problem results simplified since two first integrals of motion are present (the energy and the symmetry). However the scheme works if a suitable minisuperspace has been *a priori* defined. Here we used FRW minisuperspaces.

Some final remarks are necessary at this point. First of all, by this technique, one is capable of selecting higher–order terms as $R^{3/2}$ or $\sqrt{R\Box R}$ of physical interest since they can be connected to the one–loop or trace anomaly corrections of the effective action of quantum gravity. It is worthwhile to stress that such terms are not perturbatively introduced but emerge by the request of symmetry. Furthermore, the system of partial differential equations (3.6)–(3.12) can have several solutions and their finding out is just a question of mathematical ability. In this paper, we have not made an exhaustive list of the possible Noether symmetries in higher–order theories, but we have only presented some examples in fourth-, sixth-, and eighth–order models.

A further point which has to be stressed is that obtaining some higher–derivative terms in the effective Lagrangian of gravity does not, automatically, makes the theory renormalizable. In other words, functions of R and $\Box R$ alone do not give renormalizable Lagrangians. For renormalization in fourth–order gravity, see e.g. [7]. Our point of view is that the existence of a Noether symmetry for the Lagrangian seems to be connected to the existence of corrective terms which one needs for renormalization and the approach presented in this paper seems a scheme which could be generalized.

Finally, the existence of a Noether symmetry makes the analysis of a given cosmology more tractable; however the existence of such a symmetry is not, in itself, a sufficient motivation for preferring a particular theory. The situation, as also shown elsewhere [22, 23, 24], becomes interesting if the selected theory is physically relevant *per se* and the Noether approach selects just it.

A further step which the authors are going to achieve is to apply the Noether Symmetry Approach to the full field theory without reducing to particular minisuperspaces as done here.

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Table I– Symmetries in Fourth–Order Models

$F(R)$	α	β
$R+2\Lambda$	0	$\beta(a,R)$
$F_0R^{3/2}$	$\beta_0 a^{-1}$	$-2\beta_0 a^{-2}R$
$F_0R+F_1R^2$	0	$\beta_0 a^{-1}$
$\sum_{j=0}^{N} F_jR^j$	0	$\beta_0(aF'')^{-1}$

Table II– Symmetries in Higher than Fourth–Order Models

$F(R,\Box R)$	α	β	γ
$F_1R+F_2(\Box R)^n(n\neq 1)$	0	β_0	0
$F_1R+F_2R^2+F_3\Box R$	0	$\beta_0 a^{-1}$	$\gamma(a,R,\Box R)$
$F_0R+F_1R^2+F_2R\Box R$	0	0	$\gamma_0 a^{-1}$
$F_2\sqrt{R(\Box R)}$	0	β_0	$\beta_0 R^{-1}\Box R$
$F_1R+F_2\sqrt{R(\Box R)}$	0	β_0	$\beta_0 R^{-1}\Box R$